Chapter 12 Online Appendix: The Mathematics of Mixed Strategies in Game Theory

In this chapter, we learned that it is sometimes best for players in games to choose their actions randomly from the set of available pure strategies. Such a strategy is called a **mixed strategy.** In this appendix, we solve for mixed strategy equilibria. As a starting point, recall the example in Table 12.5 in the text about penalty kicks in soccer.

			Goalie	
			q Left	$1-q$ Right
Kicker	p	Left	0 , 1	1 , 0
	$1-p$	Right	1 , 0	0 , 1

Note that in the text example there was no pure-strategy Nash equilibrium (no box with two checks in it using the check method). Here, however, we will identify a mixed-strategy equilibrium by allowing for probabilistic selection of strategies; that is, we will assume that the kicker plays left with probability p and the goalie plays left with probability q. Because probabilities must sum to 1 to be valid, we know that the probability the kicker plays right is $1 - p$ and the probability the goalie plays right is $1 - q$.

How can we use this information to find a mixed-strategy equilibrium? Recall that a mixed strategy is one in which a player randomizes her actions. The chapter notes that if players select probabilities appropriately, they do not prefer one strategy over the other. If the expected outcomes are *not* the same for a player, that player would have an incentive to deviate from her current strategy by selecting one of the pure strategies alone or a different mix of these strategies. Remember that the existence of this incentive indicates that the proposed set of probabilities was not truly an equilibrium to start with. These conditions, in which each player is indifferent between the strategies available to her, represent mutual best responses that can be used to derive the mixed-strategy equilibrium.

Consider the case of the kicker. From the kicker's perspective, her value depends on what the goalie does. If she plays left, she will receive a payoff of 0 with probability q (i.e., if the goalie goes left) and she will receive 1 with probability $1 - q$ (i.e., if the goalie goes right). She can then calculate her expected value from playing a particular strategy as a weighted sum of possible payoffs with the relevant probabilities serving as weights. The kicker's expected value of playing left then is

$$q(0) + (1 - q)(1) = 1 - q$$

Likewise, her expected value from playing right is

$$q(1) + (1 - q)(0) = q$$

To keep the kicker indifferent, the expected value from each of her available strategies must be equal. Therefore, $1 - q = q$ or $2q = 1$, so $q = \frac{1}{2}$.

From the goalie's perspective, her value depends on what the kicker does. If the goalie plays left, she will receive a payoff of 1 with probability p (i.e., if the kicker goes left) and she will receive 0 with probability $1 - p$ (i.e., if the kicker goes right). The goalie's expected value from playing left therefore is $p(1) + (1 - p)(0) = p$. Likewise, her expected value from playing right is $p(0) + (1 - p)(1) = 1 - p$. In equilibrium, the goalie, too, will be indifferent between the actions she randomizes over. Therefore, $p = 1 - p$ or $2p = 1$, so $p = \frac{1}{2}$. Now, we have solved for the equilibrium. Both the kicker and the goalie should randomize such that they play left exactly one half of the time and play right the other half of the time.

In this example, there was not a pure-strategy equilibrium but there was a mixed-strategy equilibrium. In other games, however, there may be both pure-strategy and mixed-strategy equilibria. Recall the example of DreamWorks and Disney in the textbook. The table below re-creates Table 12.4 from the text but adds the probabilities that the firms use to determine their randomization. Recall by the check method there are two equilibria in pure strategies that correspond to the mix-match cases (DreamWorks in May and Disney in December, and vice versa). Now, let's see if there is an equilibrium in mixed strategies. The method is the same as in the case of the kicker and the goalie above. Let's start from DreamWorks' perspective. DreamWorks knows that by opening in May, it will receive 50 if Disney also opens in May and 300 if Disney waits until December. DreamWorks' expected value from opening in May therefore is $50q + 300(1 - q)$, and its expected value from opening in December is $200q + 0(1 - q)$. DreamWorks' equilibrium condition therefore is that $50q + 300(1 - q) = 200q + 0(1 - q)$. This reduces to $300 - 250q = 200q$ or $450q = 300$, so $q = \frac{2}{3}$.

			Disney	
			q	$1 - q$
			May	December
DreamWorks	p	May	50 , 50	300 , 200
	$1 - p$	December	200 , 300	0 , 0

*Payoffs are measured in millions of dollars of profit.

From Disney's perspective, the calculations are similar. The expected value from opening in May is $50p + 300(1 - p)$ and from opening in December is $200p + 0(1 - p)$. The equilibrium condition therefore is symmetric to DreamWorks': $50p + 300(1 - p) = 200p + 0(1 - p)$ and so $p = \frac{2}{3}$. Thus, there is a third Nash equilibrium in the release date game: Each company opens in May with probability $\frac{2}{3}$ and in December with probability $\frac{1}{3}$.

The kicker–goalie and DreamWorks–Disney examples are cases of symmetric games in which the strategies and payoffs are the same across players. We can also solve for both pure- and mixed-strategy Nash equilibria in nonsymmetric games in which the strategies and payoffs are not the same across players. The following Figure It Out exercise provides an example.

12OA.1 figure it out

Suppose that Sally and Suzanne are selling jewelry at a local flea market. On a given Saturday, Sally can choose to either bring earrings or bracelets and Suzanne can choose to display rings or necklaces. Demands for their products are interrelated because jewelry customers at flea markets are searching for pieces that look nice together even if they come from different vendors. Sally and Suzanne's profits, given their respective strategies, are shown in the table on the right:

		Suzanne	
		Rings	Necklaces
Sally	Earrings	100, 70	30, 40
	Bracelets	20, 10	80, 90

a. What are the pure-strategy Nash equilibria, if any?
b. What is the mixed-strategy Nash equilibrium?

Solution:

a. The check method leads us to find two pure-strategy Nash equilibria. These correspond to Sally bringing earrings while Suzanne brings rings, and to Sally offering bracelets while Suzanne offers necklaces.

			Suzanne	
			q	$1 - q$
			Rings	Necklaces
Sally	p	Earrings	✓ 100, 70 ✓	30, 40
	$1 - p$	Bracelets	20, 10	✓ 80, 90 ✓

b. To check for a mixed-strategy Nash equilibrium, we can assign probability p that Sally brings earrings and probability q that Suzanne displays rings. Accordingly, the probability that Sally brings bracelets is $(1 - p)$, and the probability that Suzanne offers necklaces is $(1 - q)$.

			Suzanne	
			q	$1 - q$
			Rings	Necklaces
Sally	p	Earrings	100, 70	30, 40
	$1 - p$	Bracelets	20, 10	80, 90

Now, we can set up the equilibrium conditions. For Sally, the expected value of earrings must equal the expected value of bracelets. For Suzanne, the expected value of rings must equal the expected value of necklaces. Note that Sally's expected value depends on the probabilities that Suzanne brings either rings or necklaces, and Suzanne's expected value depends on the probabilities that Sally offers earrings or bracelets.

For Sally:

$$100q + 30(1 - q) = 20q + 80(1 - q)$$
$$70q + 30 = 80 - 60q$$
$$130q = 50$$
$$q = \frac{5}{13}$$

For Suzanne:

$$70p + 10(1 - p) = 40p + 90(1 - p)$$
$$60p + 10 = 90 - 50p$$
$$110p = 80$$
$$p = \frac{8}{11}$$

The mixed-strategy Nash equilibrium therefore is for Sally to bring earrings with probability $\frac{8}{11}$ (because p was the probability assigned to earrings) and to display bracelets with probability $\frac{3}{11}$ (because $1 - p$ was the probability assigned to bracelets), and for Suzanne to sell rings with probability $\frac{5}{13}$ and necklaces with probability $\frac{8}{13}$ (because these q and $1 - q$ probabilities were assigned to rings and necklaces, respectively).

Problems

1. Suppose that two players (let's call them A and B) play a game where player A has the opportunity to move a game piece up or down, while player B can choose left or right. Payoffs for this game are given in the table below:

A \ B	Left	Right
Up	45, 55	15, 35
Down	5, 25	95, 65

   a. What are the pure-strategy Nash equilibria, if any?
   b. What is the mixed-strategy Nash equilibrium?

2. Imagine twin sisters Matilda and Muriel who are selecting their birthday presents. They can each ask their parents for a book or shoes (and their parents always buy them what they ask for). Suppose that each sister only cares about whether or not she is receiving the same present as the other (as opposed to which present she herself receives). Particularly, Matilda values getting something different from her sister and dislikes when they receive the same thing. Muriel, on the other hand, wants to receive the same thing as her sister and dislikes when they receive different presents. Payoffs are as given in the table below:

Matilda \ Muriel	Book	Shoes
Book	–2, 2	2, –2
Shoes	2, –2	–2, 2

   a. What are the pure-strategy Nash equilibria, if any?
   b. What is the mixed-strategy Nash equilibrium?