

# Chapter 3 Online Appendix: The Calculus of Consumer and Producer Surplus

Just as we can use calculus to analyze the slopes of supply and demand curves, we can apply calculus to find the area under the curves, or consumer and producer surplus. Specifically, this online appendix uses integration to find these areas. In the text, we were only able to solve for consumer and producer surpluses in cases in which the supply and demand curves were linear. What's powerful about integration is that it can solve for consumer and producer surplus for linear and nonlinear curves. We begin with a brief review of integration techniques before diving into the specific economic applications of integrals.

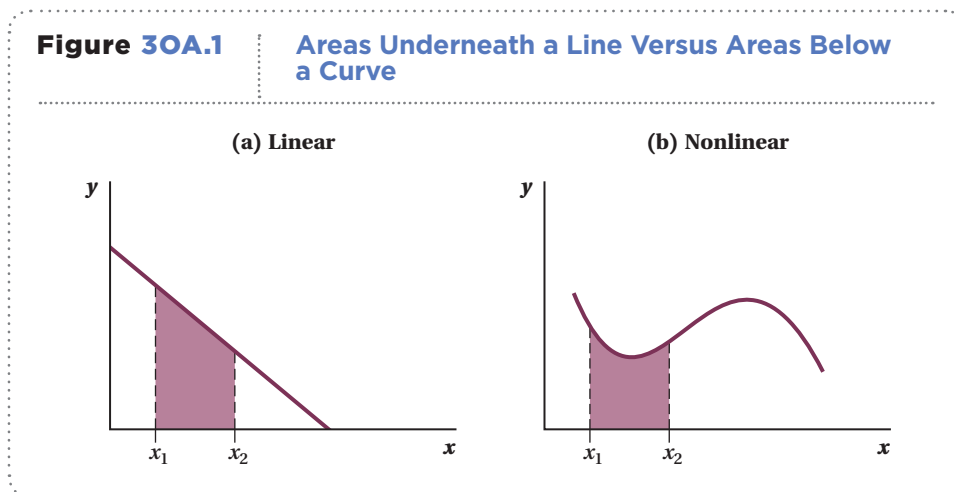
## Review of Integration Methods from Calculus

Integrals are related to the derivatives we reviewed in the book's mathematical appendix in that integration and differentiation are inverses of each other. If we integrate a derivative, we get back the original function, and vice versa, if we differentiate an integral, we also get back the original function<sup>1</sup>. The operations therefore undo each other.

It's easiest to review integration by starting with an example. Given a curve of the form  $y = f(x)$ , we can solve for the area under the curve<sup>2</sup> between the values  $x_1$  and  $x_2$ . Conventionally, we write this integral as

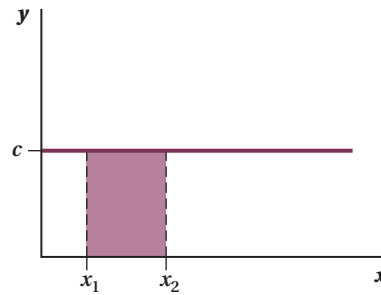
$$\int_{x_1}^{x_2} f(x) dx$$

Depending on the exact functional form, there are some basic rules for solving this integral. As in the review of derivatives in this book's Math Review Appendix, this is not a comprehensive list of rules. Instead, these are the rules that are most relevant for the topics presented in this chapter. Figure 3OA.1 provides examples of a linear (panel a) and nonlinear (panel b) functional form. In both cases, integration could be used to calculate the shaded areas.



<sup>1</sup> An exception is if there are constants of integration as you may remember from your calculus class.

<sup>2</sup> Note that we are focusing on definite integrals here, which are calculated over a specified region of a function, because these are most relevant to the economic applications we'll be looking at.

**Figure 30A.2** Integral of a Constant

**Integral of a Constant** If  $f(x) = c$ , where  $c$  is some constant, then we can solve the integral  $\int_{x_1}^{x_2} c \, dx = [cx]_{x_1}^{x_2} = cx_2 - cx_1$ . This result hinges on the fact that  $f(x) = c$  is a horizontal line at  $y = c$ , as in Figure 30A.2. As a consequence, the area under a horizontal line between  $x_1$  and  $x_2$  forms a rectangle. We know that the area of a rectangle is its base (here, the distance  $x_2$  minus  $x_1$ ) times its height ( $c - 0 = c$ ). So, the area is  $(x_2 - x_1) \times c = cx_2 - cx_1$ .

**Power Rule for Integrals** Now take  $f(x) = cx^n$ , where  $c$  is again some constant. This type of curvature is common in economics, even in the shapes of some supply and demand curves. The integral of this

$$\text{function is } \int_{x_1}^{x_2} cx^n \, dx = \left[ \frac{cx^{n+1}}{n+1} \right]_{x_1}^{x_2} = \frac{cx_2^{n+1}}{n+1} - \frac{cx_1^{n+1}}{n+1}.$$

**Addition and Subtraction Rules for Integrals** Suppose  $f(x) = u(x) + v(x)$ . Then the integral is intuitively easy to find: Just as with derivatives, the integral of an additive function is the integral of the first term plus the integral of the second term:

$$\int_{x_1}^{x_2} (u(x) + v(x)) \, dx = \int_{x_1}^{x_2} u(x) \, dx + \int_{x_1}^{x_2} v(x) \, dx$$

Similarly, if  $f(x) = u(x) - v(x)$ , the integral

$$\int_{x_1}^{x_2} (u(x) - v(x)) \, dx = \int_{x_1}^{x_2} u(x) \, dx - \int_{x_1}^{x_2} v(x) \, dx$$

## Consumer Surplus Using Calculus

**Consumer surplus** is the difference between the amount consumers would be willing to pay for a good or service and the amount they actually have to pay. Recall that we can think of total consumer surplus as the sum of differences between willingness to pay and market price at each quantity. In the chapter, we summed the area under the curve at discrete quantities (e.g., at quantities 0, 1, 2, 3, etc.). This method works for determining consumer surplus under a line, but it's not exact for nonlinear curves when geometry cannot be used to determine areas. To provide an accurate measure of consumer surplus under a nonlinear demand curve, we need to be able to measure the difference between a consumer's willingness to pay and the price at *every* point along the curve. This means we want to consider much finer fractional units than we are able to with geometric methods. This is where integration comes into play: Integration allows us to think about—and more importantly, *measure*—the total consumer surplus under a curve.

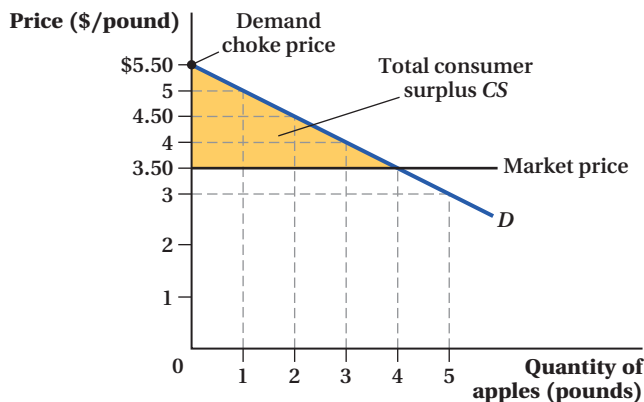
The consumer surplus calculation for the equilibrium case can be written using integral calculus as the following:

$$CS = \int_0^{Q_e} (P(Q^D) - P_e) dQ$$

where  $Q_e$  and  $P_e$  are the equilibrium quantity and price, respectively, and  $P(Q^D)$  is the inverse demand curve. Recall that the inverse demand curve is a demand curve written in the form of price ( $P$ ) as a function of quantity demanded ( $Q^D$ ).

Note that we are evaluating the function  $P(Q^D) - P_e$ . Had we wanted the entire area under the demand curve, we would simply integrate  $P(Q^D)$  by itself. This integral would then give us the total benefit of  $Q$  units of the good. But, this fails to take into account the cost of the good, so we instead want the area between demand  $P(Q^D)$  and the equilibrium price  $P_e$ . We've also indexed this integral over the range  $(0, Q_e)$ . Why? Consumer surplus is the sum of this difference over quantities that are traded, so we will evaluate the integral between 0 and  $Q_e$ .

For an example, Figure 3.1 is reproduced from the text. Note that the demand curve illustrated is linear. In slope-intercept (or “inverse” demand) form,<sup>3</sup> the equation of this line is  $P = 5.5 - 0.5Q^D$ . The consumer surplus when the market price is \$3.50 equals the area of the shaded triangle. Using the formula for the area of a triangle from geometry, this is one half times the base times the height:  $0.5 \times (4 - 0) \times (\$5.50 - \$3.50) = \$4$ .



Let's think about how we can calculate this same area using calculus. As before, we want to calculate the area under the demand curve between the quantities 0 and 4, and the prices \$3.50 and \$5.50. Using the rules of integration and calling  $Q^D$  just  $Q$  after plugging the demand curve in, the consumer surplus can be calculated as

$$CS = \int_0^{Q_e} (P(Q^D) - P_e) dQ = \int_0^4 ((5.5 - 0.5Q) - 3.5) dQ = \int_0^4 (2 - 0.5Q) dQ$$

Note that we have substituted the inverse demand curve minus the equilibrium price into the consumer surplus formula, and because we are integrating over the quantity axis, we will evaluate this integral between the quantity values 0 and 4. To calculate this integral, recall that we can use the addition and subtraction rule of integrals to divide the calculation into two parts. We can then use the integral of a constant rule on the first part and the power rule for integrals on the second part. Specifically,

$$\begin{aligned} CS &= \int_0^4 (2 - 0.5Q) dQ \\ &= \int_0^4 2 dQ - \int_0^4 0.5Q dQ \\ &= [2Q]_0^4 - \left[ \frac{0.5Q^2}{2} \right]_0^4 \\ &= [2(4) - 2(0)] - \left[ \frac{0.5(4)^2}{2} - \frac{0.5(0)^2}{2} \right] = (8 - 0) - (4 - 0) = 4 \end{aligned}$$

<sup>3</sup> A review of how to compute this equation appears in the Math Review Appendix at the end of your book.

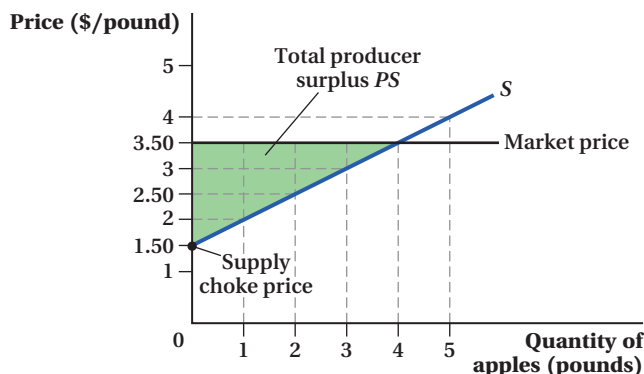
Note that this is the same \$4 consumer surplus value that we found earlier using geometry.

## Producer Surplus Using Calculus

**Producer surplus** is the difference between the price producers actually receive and the lowest price for which they would have been willing to sell their good or service. As in the case of consumer surplus, we can calculate the producer surplus area using integral calculus. Particularly, for the equilibrium case, we want to measure the area below equilibrium price and above the supply curve. This can be calculated as

$$PS = \int_0^{Q_e} (P_e - P(Q^S)) dQ$$

where  $Q_e$  and  $P_e$  are again the equilibrium quantity and price, respectively, and  $P(Q^S)$  is the inverse supply curve. Recall that the inverse supply curve<sup>4</sup> is a supply curve written in the form of price ( $P$ ) as a function of quantity supplied ( $Q^S$ ). Because producer surplus is the difference between market price and the supply price, we want to include only the area between the equilibrium price  $P_e$  and the supply curve  $P(Q^S)$  and therefore we take the difference between these when setting up the integral. Because the integration again occurs over the quantity axis, we evaluate the integral between 0 and  $Q_e$ .



Here, Figure 3.2 is reproduced from the text. In slope-intercept form, the linear supply curve depicted is  $P = 1.5 + 0.5Q^S$ . Let's calculate the producer surplus using the integration techniques we reviewed above. In particular, we want to find the area under the equilibrium price and above the supply curve between the quantities 0 and 4. To do so, evaluate the integral of the equilibrium price minus the inverse supply curve over the range 0 to 4 along the quantity axis<sup>5</sup>:

<sup>4</sup> The inverse supply function can be thought of as describing marginal cost, the way that inverse demand can be thought of as capturing marginal benefit. By thinking of inverse supply and demand as representing marginal benefits and costs at the aggregate level, it makes sense to think of the mathematical concept of integration as a way to arrive at measures of total benefits and costs of market transactions across individual consumers and producers.

<sup>5</sup> Note that the integral calculation for producer surplus in this case works out to be equivalent to consumer surplus and therefore we know that the answer to this integral is \$4 worth of producer surplus. The symmetry between consumer and producer surplus in this example relates to a feature of this particular problem. Note that the elasticities of supply and demand in absolute value form are equal for the supply and demand curves illustrated in Figures 3.1 and 3.2 in the text, which are used here. For cases in which these elasticities (and therefore the slopes of supply and demand) are different in absolute value however, consumer and producer surplus can take on extremely different values. Figure 3.3 and the related discussion in the text elaborate on how elasticities matter for surplus calculations.

$$\begin{aligned}
 PS &= \int_0^{Q^e} (P_e - P(Q^S)) dQ \\
 &= \int_0^4 (3.5 - (1.5 + 0.5Q)) dQ = \int_0^4 (2 - 0.5Q) dQ \\
 &= \int_0^4 2 dQ - \int_0^4 0.5Q dQ \\
 &= [2Q]_0^4 - \left[ \frac{0.5Q^2}{2} \right]_0^4 \\
 &= [2(4) - 2(0)] - \left[ \frac{0.5(4)^2}{2} - \frac{0.5(0)^2}{2} \right] \\
 &= (8 - 0) - (4 - 0) = 4
 \end{aligned}$$

Because the supply curve is linear, we can easily use geometry to confirm our solution to the producer surplus. In particular, producer surplus is the area of the triangle, or one half times the base times the height of the shaded triangle:  $0.5 \times (4 - 0) \times (\$3.50 - \$1.50) = \$4$ .

It's easy to calculate the consumer and producer surpluses of this function using geometric areas. So, why did we bother with integration? As you'll see in the Figure It Out exercise that follows, integration is a method that applies to more than the specific case of linear functions. Because of this, using integrals is a universal method that we can apply to both linear and nonlinear functions.

### 30A.1 figure it out

Reconsider Figure It Out 3.1 from the text. Now, think about daily newspapers in a second smaller Midwestern city. Suppose that supply is  $P = 100 + 0.02Q^2$  and the demand curve for daily newspapers in this second city is given by  $P = 200 - 0.02Q^2$ , where quantities supplied and demanded are measured in thousands of newspapers per day, and price in this case is in cents per newspaper. Note that these supply and demand curves are nonlinear. Also note that they are already written in inverse form with prices as a function of quantity supplied and demanded, respectively.

- Find the equilibrium price and quantity.
- Calculate the consumer and producer surplus at the equilibrium price.

#### Solution:

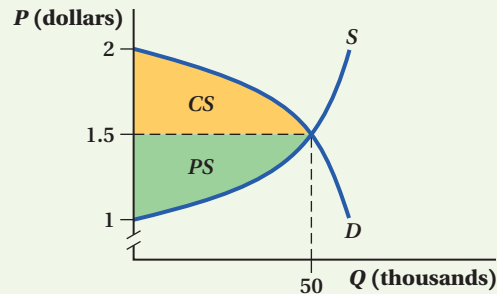
a. Equilibrium still occurs at the intersection of supply and demand even when curves are nonlinear. Because we already have these in inverse form, we can set prices equal instead of quantities as a place to start. Doing this, we see that  $200 - 0.02Q^2 = 100 + 0.02Q^2$ . This means that  $100 = 0.04Q^2$  or  $2,500 = Q^2$  or  $Q = 50$ . Since this is measured in thousands, this corresponds to 50,000 newspapers daily.<sup>6</sup> We can calculate the price of a newspaper by substituting the quantity of 50 into either the supply or demand equation (or both to double-check). Doing this for the demand side, we see that  $P = 200 - 0.02(50)^2 = 150$ . For the supply side, we can confirm that  $P = 100 + 0.02(50)^2 = 150$ . Since price is in cents, this means that a newspaper costs \$1.50 in the given market.

<sup>6</sup> Note that the equation technically has a second solution at  $-50$ . Quantities, however, in economics are non-negative by definition, and therefore we rule out this one.

b. The figure below illustrates the consumer and producer surplus areas of interest where price is translated to dollars and quantity is still in thousands. Using the integral equations for consumer and producer surplus and substituting the inverse supply and demand equations as given and the equilibrium quantity and price as calculated in part (a), we can see that

$$\begin{aligned}
 CS &= \int_0^{50} ((200 - 0.02Q^2) - 150) dQ = \int_0^{50} (50 - 0.02Q^2) dQ \\
 &= \int_0^{50} 50 dQ - \int_0^{50} 0.02Q^2 dQ \\
 &= [50Q]_0^{50} - \left[ \frac{0.02Q^3}{3} \right]_0^{50} \\
 &= [50(50) - 50(0)] - \left[ \frac{0.02(50)^3}{3} - \frac{0.02(0)^3}{3} \right] \\
 &= (2,500 - 0) - (833.33 - 0) = 1,666.67 \\
 PS &= \int_0^{50} (150 - (100 + 0.02Q^2)) dQ = \int_0^{50} (50 - 0.02Q^2) dQ = 1,666.67
 \end{aligned}$$

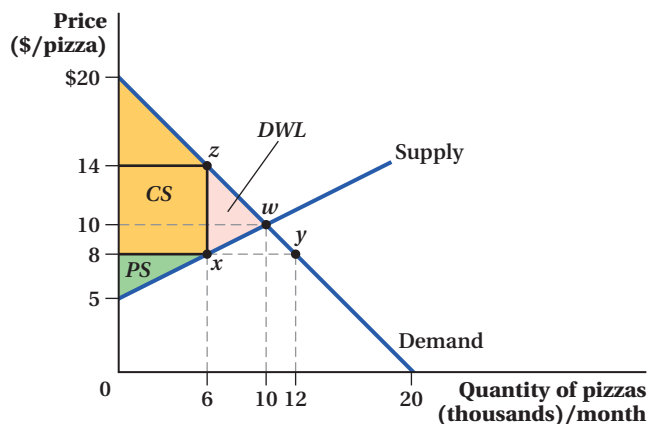
Because prices are denominated in cents, the consumer and producer surplus areas can be interpreted as approximately \$16.67 each. The fact that these work out to the same number is a coincidence of the particular shapes of the supply and demand curves in this problem. The example in the next section illustrates a case in which consumer and producer surplus values are quite different.



## Distributional Effects of Intervention Using Calculus

So far we have calculated consumer and producer surpluses in a static equilibrium. However, integration methods also allow us to measure the effects of policy changes, such as the price or quantity controls described in Chapter 3, on consumer and producer surpluses. We can also take the integration one step further and use it to determine the deadweight loss of an intervention.

The basic integration formulas do not change from those we've discussed thus far. A couple of aspects of what we evaluate may change, however. First, the integral may be evaluated over a different range than over the quantities from 0 to the equilibrium quantity; for instance, you may be interested in evaluating the area over the range from 0 to a quota level. In addition, the area of interest may be between the supply or demand curve and a price other than the equilibrium price, such as a legislated price set under a price floor or ceiling or a price determined by the market after a quantity restriction (quota) is in place.



Let's reconsider the supply and demand curves for pizzas measured in pizzas per month and dollars that are illustrated in Figure 3.7 in the text:  $Q^D = 20,000 - 1,000P$  and  $Q^S = 2,000P - 10,000$ , and the intervention of a price ceiling of \$8 per pizza. Suppose that we are interested in calculating consumer surplus, producer surplus, and deadweight loss after the intervention using calculus. As calculated in the text, at a price of \$8, quantity traded is 6,000. We can therefore calculate consumer surplus as the area under the inverse demand curve ( $P = 20 - 0.001Q$ ) and above the price of \$8 between 0 and 6,000 on the quantity axis:

$$\begin{aligned}
 CS &= \int_0^{6,000} (20 - 0.001Q - 8) dQ = \int_0^{6,000} (12 - 0.001Q) dQ \\
 &= \int_0^{6,000} 12 dQ - \int_0^{6,000} 0.001Q dQ \\
 &= [12Q]_0^{6,000} - \left[ \frac{0.001Q^2}{2} \right]_0^{6,000} \\
 &= [12(6,000) - 12(0)] - \left[ \frac{0.001(6,000)^2}{2} - \frac{0.001(0)^2}{2} \right] \\
 &= (72,000 - 0) - (18,000 - 0) = 54,000
 \end{aligned}$$

Note that this is the same \$54,000 per month consumer surplus after the price ceiling that is found utilizing geometry in the text. The analysis using calculus for producer surplus is similar. Now, we can calculate the area under the price of \$8 and above the inverse supply curve ( $P = 5 + 0.0005Q$ ) between 0 and 6,000 quantity:

$$\begin{aligned}
 PS &= \int_0^{6,000} (8 - (5 + 0.0005Q)) dQ = \int_0^{6,000} (3 - 0.0005Q) dQ \\
 &= \int_0^{6,000} 3 dQ - \int_0^{6,000} 0.0005Q dQ \\
 &= [3Q]_0^{6,000} - \left[ \frac{0.0005Q^2}{2} \right]_0^{6,000} \\
 &= [3(6,000) - 3(0)] - \left[ \frac{0.0005(6,000)^2}{2} - \frac{0.0005(0)^2}{2} \right] \\
 &= (18,000 - 0) - (9,000 - 0) = 9,000
 \end{aligned}$$

This is the same \$9,000 per month producer surplus after the price ceiling that is found in the text.<sup>7</sup>

<sup>7</sup> Note that, for this example, the absolute values of the price elasticities of supply and demand are different, and therefore the consumer and producer surplus values are not identical before or after the intervention.

Finally, we can calculate deadweight loss as the area under the inverse demand curve and above the inverse supply curve over the range of quantity that is not traded as the result of the intervention. In the free-market case, 10,000 pizzas are sold. In contrast, only 6,000 are traded after the price ceiling is implemented. We can therefore form the integral by taking the difference between the inverse demand and inverse supply equations (as opposed to between one of these and the equilibrium price, as we did for the consumer and producer surplus calculations). The deadweight loss calculation therefore is

$$\begin{aligned}
 DWL &= \int_{6,000}^{10,000} ((20 - 0.001Q) - (5 + 0.0005Q))dQ = \int_{6,000}^{10,000} (15 - 0.0015Q)dQ \\
 &= \int_{6,000}^{10,000} 15dQ - \int_{6,000}^{10,000} 0.0015QdQ \\
 &= [15Q]_{6,000}^{10,000} - \left[ \frac{0.0015Q^2}{2} \right]_{6,000}^{10,000} \\
 &= [15(10,000) - 15(6,000)] - \left[ \frac{0.0015(10,000)^2}{2} - \frac{0.0015(6,000)^2}{2} \right] \\
 &= (150,000 - 90,000) - (75,000 - 27,000) = 60,000 - 48,000 = 12,000
 \end{aligned}$$

We have confirmed that deadweight loss from the pizza price ceiling is \$12,000, as in the example in the chapter. The following Figure It Out feature presents an example of analyzing the distributional impact of a policy change when supply and demand curves are nonlinear.

### 30A.2 figure it out

Reconsider the previous Figure It Out concerning daily newspapers in a Midwestern city. The inverse demand curve for daily newspapers is again  $P = 200 - 0.02Q^2$  and inverse supply is  $P = 100 + 0.02Q^2$ . Quantities supplied and demanded are measured in thousands of newspapers per day, and price is in cents per newspaper. Suppose the local government institutes a quantity regulation (quota) on newspapers at 40,000.

- Calculate the new consumer and producer surplus values after this intervention.
- What is the deadweight loss associated with this intervention?

#### Solution:

a. The consumer surplus, producer surplus, and deadweight loss areas after the intervention are illustrated in the figure on the next page. To find the new market trading price at a quantity of 40 (in thousands), we can plug 40 into the demand equation:  $200 - 0.02(40)^2 = 168$ . The new price therefore is \$1.68. Consumer surplus is now

$$\begin{aligned}
 CS &= \int_0^{40} ((200 - 0.02Q^2) - 168)dQ = \int_0^{40} (32 - 0.02Q^2)dQ \\
 &= \int_0^{40} 32dQ - \int_0^{40} 0.02Q^2dQ \\
 &= [32Q]_0^{40} - \left[ \frac{0.02Q^3}{3} \right]_0^{40} \\
 &= [32(40) - 32(0)] - \left[ \frac{0.02(40)^3}{3} - \frac{0.02(0)^3}{3} \right] \\
 &= (1,280 - 0) - (426.67 - 0) = 853.33
 \end{aligned}$$



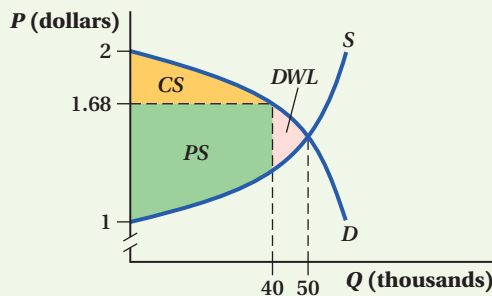
$$\begin{aligned}
 PS &= \int_0^{40} (168 - (100 + 0.02Q^2)) dQ = \int_0^{40} (68 - 0.02Q^2) dQ \\
 &= \int_0^{40} 68 dQ - \int_0^{40} 0.02Q^2 dQ \\
 &= [68Q]_0^{40} - \left[ \frac{0.02Q^3}{3} \right]_0^{40} \\
 &= [68(40) - 68(0)] - \left[ \frac{0.02(40)^3}{3} - \frac{0.02(0)^3}{3} \right] \\
 &= (2,720 - 0) - (426.67 - 0) = 2,293.33
 \end{aligned}$$

Since quantities are in thousands and prices in cents, we need to multiply the consumer and producer surplus answers by 1,000 (to account for the quantity unit of measurement) and divide by 100 (to account for the currency unit of measurement). Consumer surplus then is approximately \$8,533 and producer surplus is \$22,933.

b. To calculate deadweight loss, note from the figure that we are interested in the area under the demand curve and above the supply curve between a quantity of 40 and 50. We can therefore form the integral by taking the difference between the inverse demand and inverse supply equations (as opposed to between one of these and the equilibrium price, as we did for the consumer and producer surplus calculations). We then want to evaluate this integral between quantity levels of 40 and 50. In other words, we will evaluate this integral over the transactions that would occur in the absence of intervention but that do not occur when the quota is put in place:

$$\begin{aligned}
 DWL &= \int_{40}^{50} ((200 - 0.02Q^2) - (100 + 0.02Q^2)) dQ = \int_{40}^{50} (100 - 0.04Q^2) dQ \\
 &= \int_{40}^{50} 100 dQ - \int_{40}^{50} 0.04Q^2 dQ \\
 &= [100Q]_{40}^{50} - \left[ \frac{0.04Q^3}{3} \right]_{40}^{50} \\
 &= [100(50) - 100(40)] - \left[ \frac{0.04(50)^3}{3} - \frac{0.04(40)^3}{3} \right] \\
 &= (5,000 - 4,000) - (1,666.67 - 853.33) = 186.66
 \end{aligned}$$

Deadweight loss therefore is approximately \$1,867 in dollar terms after accounting for the units of measurement as above.



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## Problems

- Suppose that the demand curve for a particular high-end designer purse in a medium-sized city is given by  $P = 600 - 0.01Q^2$  and supply is  $P = 100 + 0.04Q^2$ , where price is in dollars and quantity is in numbers of purses.
  - Find the equilibrium price and quantity.
  - Calculate the consumer and producer surplus at the equilibrium price.
- Suppose that the supply and demand curves for the designer purse in Problem 1 are given as above. Further suppose that the wife of a local politician owns one of these purses and wants to make sure that its particular style remains relatively exclusive. She convinces her husband to institute a quantity regulation (quota) at 30.
  - Calculate the new consumer and producer surplus values after this intervention.
  - What is the deadweight loss associated with the quota?
  - Suppose the local government institutes a price regulation (floor) at \$591 instead of the quota. Calculate the new consumer and producer surplus values after this intervention.
  - What is the deadweight loss associated with the price floor?
  - Comparing your answers for the price floor in parts (c) and (d) to your solutions for the quota in parts (a) and (b), what can you deduce about the relationship between price and quantity public policy instruments in this case?
- Suppose that the demand curve for a restaurant chain's new burger is given by  $P = 10 - 2Q^3$  and supply is  $P = 5 + 0.5Q^3$ , where price is in dollars and quantity is in hundred thousands of burgers.
  - Find the equilibrium price and quantity.
  - Calculate the consumer and producer surplus at the equilibrium price.
  - Suppose that due to pressure from groups campaigning for increased availability of restaurant food for the poor, the federal government sets a price ceiling of \$5.50. Calculate the new consumer and producer surplus.
  - Calculate the deadweight loss associated with the price ceiling regulation.