

# Chapter 8 Online Appendix: The Calculus of Long-Run Competitive Equilibria

The chapter presents the long-run competitive equilibrium using graphs and intuition. In this appendix, we learn how to solve for the equilibrium mathematically.

## Long-Run Competitive Equilibrium

The long-run competitive equilibrium is defined in the chapter as the point at which market price equals the minimum average total cost and firms would gain no profits by entering the industry. This equilibrium depends both on the decisions of individual firms and on the interaction between market supply and market demand. In the chapter's graphical presentation of the long-run competitive equilibrium (e.g., in Figures 8.16, 8.17, and 8.18), there is one graph for the market and another for the firm.

To solve for the long-run competitive equilibrium, we have to represent the firm's decision and the market interaction mathematically. From the firm's side, we need to know the long-run total cost functions for all the firms. Then we can find the minimum of the average total cost curve. We know from Chapter 7 that average total cost is minimized where the marginal cost curve crosses the average total cost curve. From Chapter 8, we know that the perfectly competitive firm maximizes profit where  $P = MC$ . Also, in the long-run equilibrium price equals average total cost. Therefore, the long-run competitive equilibrium price must equal both marginal cost and average total cost. The market demand tells us how much consumers want to buy at that equilibrium price. That's the market equilibrium quantity. Then we can divide the market equilibrium quantity by the equilibrium quantity for each firm to determine the number of firms.

Let's work through an example to make this general process more concrete. Suppose that the perfectly competitive market for corn is made up of identical firms. If the long-run total cost curve for each firm is given by  $TC = 300Q - 10Q^2 + 0.5Q^3$  (where  $Q$  is tons of corn), then in the long run, the marginal cost and average cost for each firm are

$$MC = \frac{dTC}{dQ} = \frac{d(300Q - 10Q^2 + 0.5Q^3)}{dQ} = 300 - 20Q + 1.5Q^2$$
$$ATC = \frac{TC}{Q} = \frac{300Q - 10Q^2 + 0.5Q^3}{Q} = 300 - 10Q + 0.5Q^2$$

If there are free entry and exit, the long-run equilibrium price will be equal to the minimum of average total cost and marginal cost. We can find the quantity where average total cost is minimized by setting  $MC = ATC$  and solving for  $Q$ :

$$P = MC = ATC$$
$$300 - 20Q + 1.5Q^2 = 300 - 10Q + 0.5Q^2$$
$$1.5Q^2 - 0.5Q^2 = -10Q + 20Q$$
$$Q^2 = 10Q$$
$$Q = 10$$

Alternatively, we could find the quantity where  $ATC$  is minimized by taking its first-order condition with respect to  $Q$ :

$$\frac{dATC}{dQ} = \frac{d(300 - 10Q + 0.5Q^2)}{dQ} = -10 + Q = 0$$

$$Q = 10$$

To find the price at which the firm will produce 10 units, plug this quantity into either  $MC$  or  $ATC$ . Here, we find  $P$  using the equation for marginal cost:

$$P = MC = 300 - 20(10) + 1.5(10)^2 = 300 - 200 + 150$$

$$= 250$$

It's a good idea to check that this is correct by using the average total cost equation:

$$P = ATC = 300 - 10(10) + 0.5(10)^2 = 300 - 100 + 50$$

$$= 250$$

In this example then, the price at which all the firms have zero profit and therefore no incentive to enter or leave the market is \$250. But we also need to have a market equilibrium in which the total quantity produced by all the firms equals the quantity demanded by consumers at the price of \$250. Suppose the market demand is

$$Q^D = 30,000 - 50P$$

When price is \$250,

$$Q^D = 30,000 - 50P = 30,000 - 50(250) = 30,000 - 12,500 = 17,500$$

If each firm produces 10 units, then the number of firms,  $N$ , is

$$N = \frac{\text{market output}}{\text{firm output}} = \frac{17,500}{10} = 1,750 \text{ firms}$$

In summary, this market is in a long-run competitive equilibrium when each of 1,750 identical firms produces and sells 10 units at a price of \$250.

To trace out the long-run supply curve, we observe how price and output change if demand changes. Let's add one more condition to our assumptions about identical firms and free entry and exit. Suppose that the cost curves of the firms in the corn market do not change when firms enter or exit the market. Under these assumptions, the equilibrium price will return to the same level (\$250 in our example) after demand changes, and each firm will end up producing the same output (10 units); therefore, the long-run supply curve for this market is horizontal at a price of \$250. The change in demand will result in a change in the market quantity because the number of firms will change. For example, suppose that demand decreases to

$$Q_2^D = 20,000 - 50P$$

Now the market quantity at \$250 will be

$$Q_2^D = 20,000 - 50(250) = 20,000 - 12,500 = 7,500$$

We know that each firm produces 10 units, so after the decrease in demand, the number of firms in the industry will fall from 1,750 to 750:

$$N_2 = \frac{\text{market output}}{\text{firm output}} = \frac{7,500}{10} = 750 \text{ firms}$$

## 80A.1 figure it out

Suppose that the perfectly competitive market for milk is made up of identical firms with long-run total cost functions given by  $TC(Q) = 5Q^3 - 30Q^2 + 50Q$ , where  $Q$  is a gallon of milk. Assume that these cost functions are independent of the number of firms in the market and that firms may enter or exit the market freely. Market demand is  $Q^D = 10,000 - 200P$ .

- Find the long-run equilibrium price, the quantity produced by each firm, and the number of firms in the industry.
- Suppose that market demand increases to  $Q_2^D = 14,000 - 200P$ . Solve for the new long-run competitive equilibrium.

**Solution:**

a. We know that the long-run competitive equilibrium price equals the minimum average total cost. First, find average total cost:

$$ATC = \frac{TC}{Q} = \frac{5Q^3 - 30Q^2 + 50Q}{Q} = 5Q^2 - 30Q + 50$$

Now find the quantity produced by each firm at the minimum average cost by solving the first-order condition:

$$\frac{dATC}{dQ} = \frac{d(5Q^2 - 30Q + 50)}{dQ} = 10Q - 30 = 0$$

$$10Q = 30$$

$$Q = 3$$

Then plug this quantity into the average total cost equation to get the equilibrium price:

$$P = ATC = 5(3)^2 - 30(3) + 50 = 5(9) - 90 + 50 = 5$$

At this price, the quantity demanded in the market is

$$Q^D = 10,000 - 200P = 10,000 - 200(5) = 9,000$$

So, the number of firms is

$$N = \frac{\text{market output}}{\text{firm output}} = \frac{9,000}{3} = 3,000 \text{ firms}$$

b. Because the total cost curves for the firms do not change with the number of firms, the minimum average total cost, equilibrium price, and output per firm will not change when demand changes. However, total market quantity and the number of firms will increase when demand increases. At a price of \$5, the new market quantity demanded is

$$Q_2^D = 14,000 - 200P = 14,000 - 200(5) = 14,000 - 1,000 = 13,000$$

From part (a), we know that each firm produces 3 units, so the number of firms is equal to

$$N_2 = \frac{\text{market output}}{\text{firm output}} = \frac{13,000}{3} \approx 4,333 \text{ firms}$$

## Long-Run Supply in Decreasing and Increasing Cost Industries

The horizontal long-run supply curves in the preceding examples are the result of our assumption that the cost curves of firms are not affected by increases in industry output or by increases in the number of firms in the industry. The chapter discusses how the long-run supply curve is affected when this assumption does not hold. The long-run supply curve will be upward-sloping in industries in which firms' total costs increase when industry output increases (i.e., increasing cost industries). The long-run supply curve will be downward-sloping in decreasing cost industries.

Once again, we can supplement the discussion in the text by looking at the mathematics of industries in which costs change as industry output increases. The process for finding the long-run competitive equilibrium is the same as we used in the constant cost industries we examined in the previous examples, but now we also need to account for the fact that each firm's cost changes when the size of the industry changes. Again, working through an example will make this more concrete.

Suppose that the competitive market for wheat is made up of identical firms and that there are free entry and exit. Market demand is given by  $Q^D = 80,000 - 100P$ . Each of the firms has the following total cost function:

$$TC = 2Q^3 - 40Q^2 + RQ$$

where  $Q$  is tons of wheat produced by each firm and  $R$  is the rental rate on the farmland used to grow wheat. As firms enter the wheat market, the price of farmland is bid up. Let's represent the rental rate on farmland as

$$R = 0.1N$$

where  $N$  is the number of farms growing wheat. Therefore, the total cost function is

$$TC = 2Q^3 - 40Q^2 + 0.1NQ$$

In the long-run competitive equilibrium, firms will enter or exit until profit is equal to zero or until price is equal to minimum average total cost. Let's find the minimum of average total cost by finding the quantity where marginal cost equals average total cost. First, find  $MC$  and  $ATC$ :

$$MC = \frac{dTC}{dQ} = \frac{d(2Q^3 - 40Q^2 + 0.1NQ)}{dQ} = 6Q^2 - 80Q + 0.1N$$

$$ATC = \frac{TC}{Q} = \frac{2Q^3 - 40Q^2 + 0.1NQ}{Q} = 2Q^2 - 40Q + 0.1N$$

The long-run equilibrium price will equal both marginal cost and average total cost. Set  $MC = ATC$  and solve for the quantity that minimizes  $ATC$ :

$$P = MC = ATC$$

$$6Q^2 - 80Q + 0.1N = 2Q^2 - 40Q + 0.1N$$

$$6Q^2 - 2Q^2 = -40Q + 80Q$$

$$4Q^2 = 40Q$$

$$Q = 10$$

Once we know the quantity each firm produces (10 in our example), we can find the equilibrium price in this market by plugging the equilibrium quantity produced by each firm into the marginal cost or average cost curve:

$$P = ATC = 2Q^2 - 40Q + 0.1N = 2(10)^2 - 40(10) + 0.1N = 200 - 400 + 0.1N$$

$$= -200 + 0.1N$$

In this industry, equilibrium price is a function of the number of firms  $N$ . Therefore, we need to determine the market equilibrium before we can find the exact equilibrium price. Before we do that, let's see how the equilibrium price is affected by the increase in industry output that comes from the entry of new firms. Take the derivative of price with respect to the number of firms:

$$\frac{dP}{dN} = \frac{d(-200 + 0.1N)}{dN} = 0.1 > 0$$

The increased costs that are caused by the entry of new firms translate to higher prices as the number of firms increases.

Now let's find the equilibrium of market supply and demand to determine the number of firms and equilibrium price. We know that each firm produces 10 tons of wheat; therefore, market supply is 10 multiplied by the number of firms,  $N$ . Set the equilibrium condition:

$$\begin{aligned} Q^D &= Q^S \\ 80,000 - 100P &= 10N \\ 80,000 - 100(-200 + 0.1N) &= 10N \\ 80,000 + 20,000 - 10N &= 10N \\ 100,000 &= 20N \\ N &= 5,000 \end{aligned}$$

Then equilibrium price is

$$\begin{aligned} P &= -200 + 0.1N = -200 + 0.1(5,000) = -200 + 500 \\ &= 300 \end{aligned}$$

We can trace out long-run supply by seeing how a change in demand affects the equilibrium price and quantity. Suppose that market demand increases to

$$Q_2^D = 90,000 - 100P$$

The increase in demand will cause firms to enter the market. Average total cost for each firm is still

$$ATC = 2Q^2 - 40Q + 0.1N$$

Entry of each new firm shifts the average total cost up by a constant amount, increasing the minimum value of average total cost but not changing the quantity at which  $ATC$  is minimized. We can see this by taking the first-order condition with respect to  $Q$ :

$$\begin{aligned} \frac{dATC}{dQ} &= \frac{d(2Q^2 - 40Q + 0.1N)}{dQ} = 4Q - 40 = 0 \\ Q &= 10 \end{aligned}$$

The long-run equilibrium output of individual firms is not a function of industry output so each firm will still produce 10 units, and total quantity supplied will be  $10N$ ; however, the price will rise as firms enter. We can solve for the new market equilibrium:

$$\begin{aligned} Q^D &= Q^S \\ 90,000 - 100P &= 10N \\ 90,000 - 100(-200 + 0.1N) &= 10N \\ 90,000 + 20,000 - 10N &= 10N \\ 110,000 &= 20N \\ N &= 5,500 \end{aligned}$$

Then equilibrium price is

$$\begin{aligned} P &= -200 + 0.1N = -200 + 0.1(5,500) = -200 + 550 \\ &= 350 \end{aligned}$$

In the initial equilibrium, when there were 5,000 firms in the industry, the equilibrium price was \$300 per ton and total market output was 50,000 tons of wheat. After demand increases, and total market output and the number of firms increase, the equilibrium price increases to \$350 per ton. Therefore, the long-run supply curve is upward-sloping for this increasing cost industry. The approach for finding the long-run supply curve for a decreasing cost industry is similar and will yield a downward-sloping long-run supply curve.

## Problems

1. Suppose that the market for auto detailing in a city is perfectly competitive. The auto detailing firms are identical and have long-run cost functions given by  $TC(Q) = 10Q^3 - 100Q^2 + 300Q$ . Market demand is  $Q^D = 5,000 - 90P$ .
  - a. Derive the marginal and average cost curves for a firm in this industry.
  - b. Find the quantity at which average total cost is minimized for each firm.
  - c. Find the long-run equilibrium price in this industry.
  - d. Use market demand to find the equilibrium total industry output.
  - e. Find the equilibrium number of firms.
2. Suppose that the identical firms in a perfectly competitive market for cakes have long-run total cost functions given by  $TC(Q) = 10Q^3 - 60Q^2 + 100Q$ . Total cost is independent of the number of firms and total output in the market.
  - a. Describe the long-run supply curve for this industry.
  - b. Suppose market demand is  $Q^D = 1,000 - 40P$ . Solve for the long-run competitive equilibrium price, output per firm, and number of firms in the market.
  - c. Suppose demand decreases to  $Q^D = 800 - 40P$ . Solve for the long-run competitive equilibrium price, output per firm, and number of firms in the market.
3. Suppose that the potential customers for hair braiding in a city consider hair braiding to be identical and that the market is perfectly competitive. Hair braiding requires special skills so the supply of workers in this industry is upward-sloping, and the wages earned by hair braiders increase as the industry output increases. Firms in this market face the following total cost:
 
$$TC = Q^3 - 8Q^2 + 20Q + W$$
 where  $Q$  is the number of hair braidings and  $W$  is the daily wage paid to workers. The wage, which depends on total industry output, equals  $W = 0.1NQ$ , where  $N$  is the number of firms. Market demand is  $Q^D = 500 - 20P$ .
  - a. How does average total cost for the firm change as industry output increases? What does this relationship imply for industry's long-run supply curve?
  - b. Find the long-run equilibrium output for each firm.
  - c. How does the long-run equilibrium price change as the number of firms increases?
  - d. Find the long-run equilibrium number of firms and total industry output.
  - e. Find the long-run equilibrium price.
  - f. Suppose that demand increases to  $Q^D = 1,000 - 10P$ . Find the new long-run competitive equilibrium. Does this match your prediction about the long-run supply curve from part (a)?