MATH REVIEW

APPENDIX A

Some knowledge of algebra and trigonometry is essential for a full understanding of this book. In addition, a limited ability to deal with complex numbers and derivatives (a part of calculus) is helpful, although not entirely essential. This appendix is meant as the briefest of summaries of complex numbers and differentiation, preceded by a collection of useful formulas from trigonometry, exponentials, and logarithms. It is not meant as a textbook substitute. For a highly readable self-help book on calculus, we recommend *Quick Calculus*, by D. Kleppner and N. Ramsey, Wiley, 2nd ed., 1985.

A.1 Trigonometry, exponentials, and logarithms

Here is a collection of useful formulas:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is the solution of the quadratic equation

$$ax^2 + bx + c = 0.$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\sin 2x = 2 \sin x \cos x,$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right],$$

$$\cos x \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right],$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$e^{x+y} = e^x e^y,$$

$$e^{x-y} = e^x / e^y,$$

$$x^{a/b} = \sqrt[b]{x^a},$$

$$e^{\log_e x} = x,$$

$$\log_e(xy) = \log_e x + \log_e y,$$

$$\log_e(x/y) = \log_e x - \log_e y,$$

$$\log_e x^n = n \log_e x,$$

$$\log_e e^x = x,$$

$$\log_e x = \log_e 10 \log_{10} x \approx 2.3 \log_{10} x,$$

$$a^x = e^{x \log_e a}.$$

A.2 Complex numbers

A complex number is an object of the form

$$\mathrm{N} = a + ib,$$

where a and b are real numbers and i is the square root of -1; a is called the real part, and b is called the imaginary part.[1] Boldface letters or squiggly underlines are sometimes used to denote complex numbers. At other times you're just supposed to *know*!

Complex numbers can be added, subtracted, multiplied, etc., just as real numbers:

$$(a+ib) + (c+id) = (a+c) + i(b+d),$$

$$(a+ib) - (c+id) = (a-c) + i(b-d),$$

$$(a+ib)(c+id) = (ac-bd) + i(bc+ad),$$

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i.$$

[1] Electrical engineers depart from the universal convention of $i \equiv \sqrt{-1}$, using instead the symbol j in order to avoid duplicating the use of the symbol i (which designates small-signal current). We follow the EEs in this book, but not in this Math appendix. Were we to do so, we would likely be disowned by our math colleagues.

All these operations are natural, in the sense that you just treat i as something that multiplies the imaginary part, and go ahead with ordinary arithmetic. Note that $i^2 = -1$ (used in the multiplication example) and that division is simplified by multiplying top and bottom by the *complex conjugate*, the number you get by changing the sign of the imaginary part. The complex conjugate is sometimes indicated with an asterisk. If

$$\mathbf{N} = a + ib,$$

then

$$\mathbf{N}^* = a - ib.$$

The magnitude (or *modulus*) of a complex number is a real number with no imaginary part:

$$|\mathbf{N}| = |a+ib| = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2+b^2},$$

i.e.,

$$|\mathbf{N}| = \sqrt{\mathbf{N}\mathbf{N}^*},$$

simply obtained by multiplying by the complex conjugate and taking the square root. The magnitude of the product (or quotient) of two complex numbers is simply the product (or quotient) of their magnitudes.

The real (or imaginary) part of a complex number is sometimes written as

real part of $\mathbf{N} = \mathscr{R}e(\mathbf{N})$,
imaginary part of $\mathbf{N} = \mathscr{I}m(\mathbf{N})$.

You get them by writing out the number in the form $a+ib$, then taking either a or b. This may involve some multiplication or division, since the complex number may be a real mess.

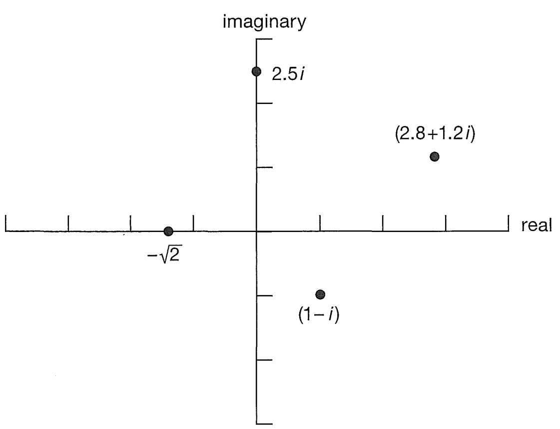

Figure A.1. Complex numbers in the "complex plane."

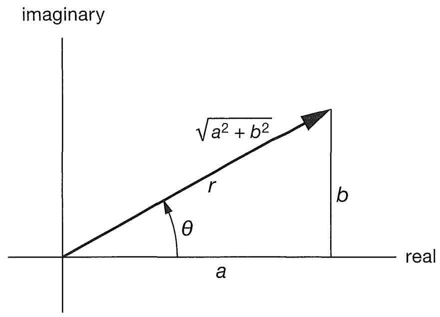

Figure A.2. Complex numbers, as magnitude and angle.

Complex numbers are sometimes represented on the complex plane. It looks just like an ordinary x, y graph, except that a complex number is represented by plotting its real part as x and its imaginary part as y, as shown in Figure A.1. In keeping with this analogy, you sometimes see complex numbers written just like x, y coordinates:

$$a + ib \leftrightarrow (a, b).$$

Just as with ordinary x, y pairs, complex numbers can be represented in polar coordinates; that's known as "magnitude, angle" representation. For example, the number $a+ib$ can also be written as (Figure A.2)

$$a + ib = r\angle\theta,$$

where[2] $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$. This is usually written in a different way, using the astonishing fact that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

(You can derive the preceding result, known as Euler's[3] formula, by expanding the exponential in a Taylor series.) Thus we have the following equivalents:

$$\begin{aligned} \mathbf{N} &= a + ib = re^{i\theta}, \\ r &= |\mathbf{N}| = \sqrt{\mathbf{N}\mathbf{N}^*} = \sqrt{a^2+b^2}, \\ \theta &= \tan^{-1}(b/a), \end{aligned}$$

i.e., the modulus r and angle θ are simply the polar coordinates of the point that represents the number in the complex plane. Polar form is handy when complex numbers have to be multiplied; you just multiply their magnitudes and add their angles (or, to divide, you divide their magnitudes and subtract their angles):

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

[2] Caution: the formula for θ returns values only between $-\pi/2$ and $+\pi/2$; the signs of both a and b, and not merely their quotient, are required for a correct value of θ in all four quadrants.

[3] Leonhard Euler, pronounced like "oiler."

Finally, to convert from polar to rectangular form, just use Euler's formula:

$$re^{i\theta} = r\cos\theta + ir\sin\theta,$$

i.e.,

$$\mathscr{Re}(re^{i\theta}) = r\cos\theta,$$
$$\mathscr{Im}(re^{i\theta}) = r\sin\theta.$$

(These can be used to easily derive the sum and difference of trigonometric functions, so you never have to remember those pesky formulas. Just work out $e^{i(x\pm y)}$.)

If you have a complex number multiplying a complex exponential, just do the necessary multiplications. If

$$\begin{aligned} \mathrm{N} &= a+ib, \\ \mathrm{N}e^{i\theta} &= (a+ib)(\cos\theta + i\sin\theta), \\ &= (a\cos\theta - b\sin\theta), \\ &\quad + i(b\cos\theta + a\sin\theta). \end{aligned}$$

When dealing with circuits and signals, the angular argument θ often takes the form of an evolving wave: $\theta = \omega t = 2\pi f t$; thus, for example, $V(t) = \mathscr{Re}(V_0 e^{i\omega t}) = V_0\cos\omega t$, etc.

A.3 Differentiation (Calculus)

We start with the concept of a *function* $f(x)$, i.e., a formula that gives a value $y = f(x)$ for each x. The function $f(x)$ should be *single valued* i.e., it should give a single value of y for each x. You can think of $y = f(x)$ as a graph, as in Figure A.3. The derivative of y with respect to x, written dy/dx ("dee y dee x"), is the *slope* of the graph of y versus x. If you draw a tangent to the curve at some point, its slope is *dy/dx at that point*; i.e., the derivative is itself a function, since it has a value at each point. In Figure A.3 the slope at the point $(1,1)$ happens to be 2, whereas the slope at the origin is zero (we'll see shortly how to compute the derivative).

In mathematical terms, the derivative is the limiting value of the ratio of the change in y (Δy) to the change in x (Δx), as Δx goes to zero. To quote a song once sung in the hallowed halls of Harvard (by Tom Lehrer and Lewis Branscomb),

> You take a function of x, and you call it y
> Take any x-nought that you care to try
> Make a little change and call it delta x
> The corresponding change in y is what you find nex'
> And then you take the quotient, and now, carefully

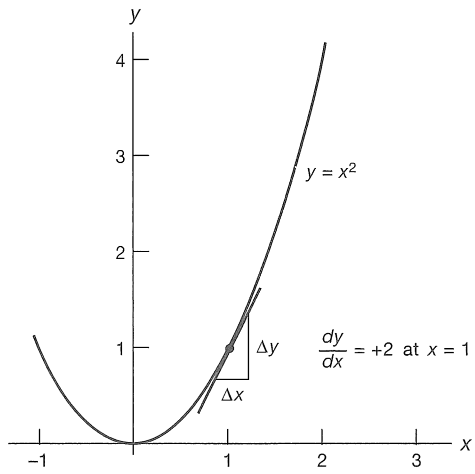

Figure A.3. A single-valued function: $f(x) = x^2$.

> Send delta x to zero, and I think you'll see
> That what the limit gives us (if our work all checks)
> Is what you call dy/dx...
> It's just dy/dx.
>
> (*The Derivative Song*, sung to the tune of *There'll Be Some Changes Made*, W. Benton Overstreet).

Differentiation is a straightforward art, and the derivatives of many common functions are tabulated in standard tables and automatically calculated in programs like Mathematica®. Here are some rules (u and v are arbitrary functions of x, and a represents a constant).

A.3.1 Derivatives of some common functions

$$\frac{d}{dx}a = 0$$

$$\frac{d}{dx}ax = a$$

$$\frac{d}{dx}ax^n = anx^{n-1},$$

$$\frac{d}{dx}\sin ax = a\cos ax,$$

$$\frac{d}{dx}\cos ax = -a\sin ax,$$

$$\frac{d}{dx}e^{ax} = ae^{ax},$$

$$\frac{d}{dx}\log_e x = 1/x.$$

A.3.2 Some rules for combining derivatives

Here $u(x)$ and $v(x)$ represent generic functions of x:

$$\frac{d}{dx}au(x) = a\frac{d}{dx}u(x),$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx}+\frac{dv}{dx},$$

$$\frac{d}{dx}uv = u\frac{d}{dx}v+v\frac{d}{dx}u,$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2},$$

$$\frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\{u[v(x)]\} = \frac{du}{dv}\frac{dv}{dx}.$$

The last one is very useful and is called the chain rule.

A.3.3 Some examples of differentiation

$$\frac{d}{dx}x^2 = 2x,$$

$$\frac{d}{dx}(1/x^{\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}},$$

$$\frac{d}{dx}xe^x = xe^x+e^x \qquad \text{(product rule)},$$

$$\frac{d}{dx}e^{-x^2} = -2xe^{-x^2} \qquad \text{(chain rule)},$$

$$\frac{d}{dx}a^x = \frac{d}{dx}(e^{x\log_e a}) = a^x\log_e a \quad \text{(chain rule)}.$$

Once you have differentiated a function, you often want to evaluate the value of the derivative at some point. Other times you may want to find a minimum or maximum of the function; that's the same thing as having a zero derivative, so you can just set the derivative equal to zero and solve for x. For example, you can easily determine that the slope of the function plotted in Figure A.3 equals 2 at x=1, and that its minimum occurs at x=0 (where its slope is zero).